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The Form of Perfect and Multiperfect Numbers

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Abstract

Originally a fascination of the Ancient Greeks, today what is known about perfect numbers is well-documented. However, there is significantly less literature regarding the general form of multiperfect numbers. Our research reveals Euler's 18th century result and Jacques Touchard's more recent result on odd perfect numbers with multiplicity 2, as a special case for multiply perfect numbers with multiplicity *k*. Additionally, this paper investigates the form of odd perfect numbers under specific cases.

Introduction

- A **Perfect Number** is any positive integer *n* whose divisors including 1 and *n* sum to twice the number *n*.
  Ex: 6 is a perfect number because 1+2+3+6 = 12 = 2(6).
- The Greek mathematician, Euclid discovered the first four perfect numbers and characterized them as $(2^{p-1})(2^p-1)$ given that *p* and (2^p-1) are both prime numbers.
- A **prime** number is an integer *p* > 1 if the only positive divisors of *p* are 1 and *p* itself.
  Ex: 2, 3, 5, 7, 11 …
- A **Mersenne prime** is a special type of prime number of the form (2^p-1) where *p* is prime.
  Ex: $2^2-1 = 3$, *p* = 2
  $2^3-1 = 7$, *p* = 3
- When working with perfect numbers it is convenient to define a function that sums up all the divisors of a number, appropriately called the **sum of divisors function.**

$$\sigma: \mathbf{N}\to\mathbf{N} \text{ defined by } \sigma(n) = \textstyle\sum_{d|n} d.$$

  Ex: $\sigma(5) = 1+5$
  $\sigma(6) = 1+2+3+6 = 12$
- Thus, a **perfect Number** is a positive integer *n* where $\sigma(n) = 2n$.
- The numbers 6 and 28 are the smallest perfect numbers.
  σ(6)=1+2+3+6=12=2(6).
  σ(28)=1+2+4+7+14+28=56=2(28).
- A **multiperfect number** (also called multiply perfect numbers, k-perfect numbers) is a positive integer *n* whose divisors including 1 and *n* sum to *k*-times the number *n*. Hence, σ(*n*) = *kn*, where *k* is an integer.
  Ex: σ(120) = 1+2+3+4+5+6+8+10+12+15+20+30+40+60+120
  = 360 = 3(120)
  In this example, *k*=3, so 120 is a 3-perfect number.
- For multiperfect numbers, *k* is called the **multiplicity**.
- Perfect numbers are a special case of multiperfect numbers, with multiplicity *k*=2.

Even Perfect Numbers

n	p_n	Perfect Number	$(2^{p-1})(2^p-1)$
1	2	6	$2(2^2-1)$
2	3	28	$2^2(2^3-1)$
3	5	496	$2^4(2^5-1)$
4	7	8128	$2^6(2^7-1)$
5	13	33550336	$2^{12}(2^{13}-1)$
6	17	8589869056	$2^{16}(2^{17}-1)$
7	19	137438691328	$2^{18}(2^{19}-1)$
8	31	2305843008139952128	$2^{30}(2^{31}-1)$
9	61	2658455991569831744654692615953842176	$2^{60}(2^{61}-1)$
10	89	191561942608236107294793378084303638130997321548169216	$2^{88}(2^{89}-1)$

Table 1: The first 10 perfect numbers with factorization in Euler’s form.

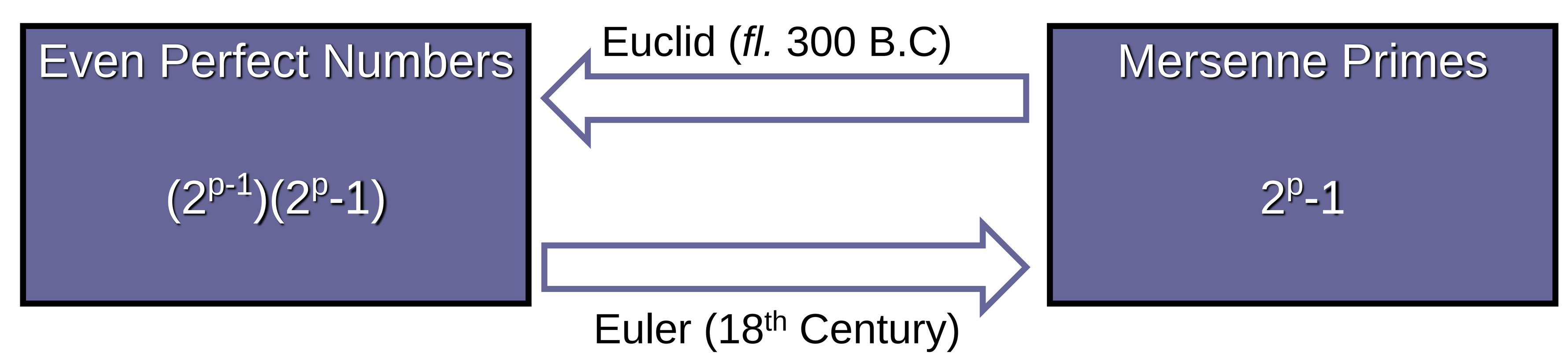

Figure 1: Euclid and Euler proved that there is a one-to-one correspondence between Mersenne primes and even perfect numbers.

-Currently there are 46 Mersenne prime numbers and thus 46 corresponding perfect numbers, all of which are even.

-Recent discovery of 45th and 46th Mersenne prime (Aug/Sept 2008) resulted in a corresponding discovery of 45th and 46th even perfect number.

Open Questions

- Are there ***infinitely*** many perfect numbers?
- Are there any ***odd perfect*** numbers?
- Are there any ***odd multiperfect*** numbers?
- Are there multiperfect numbers for ***all*** multiplicities *k*?

Odd Perfect Numbers

Euler’s Characterization of Odd Perfect Numbers:

- If an odd perfect number exists, then it is of the form

$$n=p^{\alpha}q_1^{2\beta_1}\dots q_r^{2\beta_r}$$

Where *p* and *q* are distinct primes, and $p=1+4m_1$ and $\alpha=1+4m_2$.

Touchard’s Theorem (1953):

- If an odd perfect number exists, then it is of the form
  12*m* + 1 or 36*m* + 9

Results: Odd Multiperfect Numbers

Generalization of Touchard’s Theorem

Lemma: If *n* is congruent to 2 *mod* 3, then *σ(n)* is divisible by 3.

Proof: Summing the divisors in pairs: $\sigma(n)=\sum_{d|n, d<\sqrt{n}}(d+\frac{n}{d})$.

Since $n = d\cdot(\frac{n}{d}) \equiv 2 \bmod 3$ either d ≡ 2 mod *3* and $\frac{n}{d}\equiv 1 \bmod 3$ or d ≡ *1*mod*3* and $\frac{n}{d}\equiv 2 \bmod 3$. Thus $d+(\frac{n}{d})\equiv 0 \bmod 3$ and hence *σ(n)* is divisible by 3.

Lemma: If *n* is congruent to 3 *mod* 4, then *σ(n)* is divisible by 4.

Theorem: If *n* is an odd multiperfect number of multiplicity *k*, and *k* is not divisible by 3 or 4, then 12*m* + 1 or 36*m* + 9 for some integer *m.*

Generalization of Euler’s Characterization

Theorem: Let *n* be a positive integer with unique factorization

$$n = 2^{\gamma}\prod_{i=1}^{k} p_i^{\alpha_i}\prod_{j=1}^{l} q_j^{\beta_j}$$

with $p_i \equiv 1 \bmod 4$ and $q_j \equiv 3 \bmod 4$. If at least one β_j is odd, then $4|\sigma(n)$. If all the β_j’s are even then

$$\sigma(n) = \begin{cases} \prod(\alpha_i+1) \bmod 4 & \text{if } n \text{ is even} \\ 3\prod(\alpha_i+1) \bmod 4 & \text{if } n \text{ is odd.}\end{cases}$$

Corollary: If $n \equiv 1 \bmod 4$, then $\sigma(n)\equiv\prod_{i=1}^{k}(\alpha_i+1)\bmod 4$ and if *n* is multiperfect with multiplicity *k,* then $k\equiv\prod_{i=1}^{k}(\alpha_i+1)\bmod 4$.

Theorem: If *n* is an odd multiperfect number with multiplicity *k* with 2||*k* then $n=p^{\alpha}m^2$ where *p* is prime and $p\equiv \alpha \equiv 1 \bmod 4.$

More on Perfect Numbers

Theorem: If *n* is an odd perfect number *n* not divisible by 3, then $p \equiv 1 \bmod 12$ and either $\alpha \equiv 1 \bmod 12$ or $\alpha \equiv 9 \bmod 12$ where *n* is of Euler’s form $n=p^{\alpha}m^2$.

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